# The Interpretation of Model Experiments for the Displacement of Fluids Through Porous Media

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Much of this ground has been covered adequately before. The motivation for a review of some of its detail is the conviction that the basic subject for examination, even in a discussion of scaling, is the physical mechanism of the process under study.

Even though the topic of this paper concerns some rather specific engineering problems of petroleum production it seems useful to discuss it, initially, in very general terms so that the fundamental questions might stand out more clearly.

The generalized prototype system is a relatively large body of porous and permeable material saturated with hydrocarbon fluid. This pay zone is pierced by a number of production wells through which fluid is withdrawn and by injection wells by which displacing fluids are introduced. The model experiments with which we concern ourselves are intended to enable the reservoir analyst to answer pressing economic questions such as the following: Given the existing distribution of wells, what is the optimum choice of producers and injectors, and what is the optimum set of relative production and injection rates? Considering local pumping and fluid costs, what are the relative merits of different choices of injection fluids and absolute injection rates? Would it be profitable to drill additional wells, and where should they be?

Of course, the model itself cannot directly yield the required answers; this is the responsibility of the engineer. The model experiment, however, must be designed to provide a scientific description of some well-defined aspects of the prototype behavior. The results of the experiment must be interpreted correctly to provide the background information on which the economic analysis might be made. The behavior of the model should be observed in sufficient detail to enable the interpretation to be made properly. In particular, it must be possible to define, both experimentally and theoretically, the extent to which the model is truly the analogue of the prototype. The effects of the analogy's inevitable imperfections must be evaluated in terms of the resulting differences between model and prototype behavior.

Of course any specific example of the generalized reservoir described so glibly above is encumbered by many complications. For instance, the pay zone is surrounded by other geological formations which may also be permeable and which are usually saturated by a different fluid. The convenient description of the limits of the formation of interest as boundary surfaces may thus be a poor approximation. Additional geological features of the reservoir may also be so complicated that they cannot be adequately represented in a model.

Other assumptions about the prototype reservoir, about its fluid content and about the processes going on in it, are made in order to simplify the analysis. Many such assumptions will be made here which naturally limit the range of applicability of the discussion. As far as possible, these limitations will be pointed out as they occur.

It seems worthwhile to review briefly here the requirements for a model experiment. The design of a dimensionally scaled experiment or of a pure model involves the use of the concepts of dimensional analysis or of its modern successor, the so-called *inspectional analysis*. In order that the behavior of a model and its prototype be similar there must be more than geometric similarity between them. Several other conditions also must be met. First, the partial differential equations describing the behavior must be common to both systems. Second, the required number of dimensionless parameters in the model must be contrived to have the same values as the corresponding numbers which describe the range of operation of the prototype system. Finally, the initial and boundary conditions of the systems must, when expressed in terms of the dimensionless variables, be the same. Under these stringent conditions one can expect various derived aspects of the model's behavior, such as the recovery curves from the production wells, to be similar to the corresponding descriptions of the prototype. From the results of experiments with such pure models one may calculate directly the pumping and fluid requirements as well as the expected-oil-production schedules.

The standards of purity are high, however, and are discussed below in more detail. But first one might ask whether there is any use in a less-than-perfect analogy? The major qualification for a useful semi-model is a close analogy between the equations of motion in the model and those appropriate to the prototype. Furthermore, there must exist some range of the variables over which this analogy amounts to essential equivalence. It is only in this range that useful results can be expected; all experimental details of the construction of the model, the arrangement of the boundary and initial conditions, and the means of observing its behavior, must be accomplished in such a way that the similarity of the model's equations to those of the prototype is preserved. Because of this added complication, the interpretation of an imperfect or semi-model is more critical. The predictions of prototype behavior which are made on the basis of such a model should therefore be given more cautiously.

## THE DARCY FLOW EQUATION

Before considering its more complicated variations, a few moments might usefully be spent on the unadorned Darcy equation as given, in the differential form, in elementary texts. The equation as written here

$$\nabla p = -\frac{\mu}{k}\vec{v} \tag{1}$$

neglects gravitational forces, but does not suffer from the omission as long as the fluid density is constant throughout the region of interest. If, in addition, neither the fluid viscosity $\mu$ nor the matrix permeability $k$ are functions of position, and if the fluid is incompressible, then Equation (1) may be reduced by taking the divergence to Laplace's partial differential equation in the pressure

$$\nabla^2 p = 0 \tag{2}$$

Solutions of this equation are well known. Within boundaries which are coordinate surfaces in reasonable orthogonal curvilinear coordinate systems, and if the function $p$ or the normal component of its gradient is given on the boundaries, the solution is calculable in closed form. In cases where the boundary cannot be so represented, numerical techniques are available by which $p_{(\vec{r})}$ may be evaluated to any degree of approximation at the mesh points of a grid covering the region. In addition to Darcy flow, there are several other physical processes with easily measurable dependent variables which also follow Laplace's equation. These analogous physical systems allow the construction of the various semi-models for flow through porous media.

The scaling properties of Laplace's and of the elementary form of Darcy's equation which gave rise to it are simple. To examine the conditions under which one Darcy flow system might be considered a model of another, one must determine what is required to bring such a system into similarity with a reduced system. A reduced system is described in terms of dimensionless independent variables

$$\left.\begin{aligned} \xi &= x/A \\ \eta &= y/A \\ \zeta &= z/A \end{aligned}\right\} \tag{3a}$$

where $A$ is some characteristic macroscopic distance by which a particular system is identified within the set of all geometrically similar systems. The dependent variables of the reduced system must also be given in dimensionless form. Thus, a dimensionless pressure $\hat{p}_{(\xi,\eta,\zeta)}$ is defined as the ratio of the pressure $p_{(x,y,z)}$ to some characteristic pressure drop measurable in the real system:

$$\hat{p}_{(\xi,\eta,\zeta)} = \frac{p_{(x=\xi A,\ y=\eta A,\ z=\zeta A)}}{p_i - p_o} \tag{3b}$$

Similarly a dimensionless velocity $\vec{U}$ is then defined as the ratio of the Darcy velocity $\vec{v}$ in the system to a characteristic scalar rate $W$

$$\vec{U}_{(\xi,\eta,\zeta)} = \frac{\vec{v}}{W} = \frac{\vec{v}_{(x,y,z)}\,\mu A}{k(p_i - p_o)} \tag{3c}$$

In terms of these dimensionless variables, Equations (1) and (2) are transformed into the equations of the reduced system

$$\nabla' \hat{p}_{(\xi,\eta,\zeta)} = -\vec{U} \tag{1'}$$

$$\nabla'^2 \hat{p}_{(\xi,\eta,\zeta)} = 0 \tag{2'}$$

where

$$\nabla' \equiv \left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right)$$

The boundary conditions in the reduced system are also to be derived from those in the physical system by the same transformation equations.

Note that any geometrically similar Darcy system may thus be described by the same reduced system equations if only the variables are expressed in dimensionless form. In particular, note that no scaling parameter or similarity group is involved in this case as long as $\mu$ and $k$ remain constant throughout the system. The absence of any such group makes it possible to operate dimensionally scaled models of Darcy systems of any convenient size, at any convenient rates.

The use of such modeling procedure is justified by the assumption that the unadorned differential form of Darcy's law given as Equation (1) is adequate to describe the volumetric flow velocity in the rock. For the proper consideration of the displacement of fluids within the system, a variable is also needed which describes the nature of the fluid present at any particular place. The simplest picture, though not always a very realistic one, depicts a discontinuous situation in which the fluid at any location is of either one kind or another—either the displaced liquid or the displacing—with no composition states between. A more realistic representation might include the specifications of a zone over which the nature of the fluid changes more gradually. If the composition variable is a linear measure of the quantity of a fluid component which is conserved, then the material derivative of this variable will vanish. The equation describing the rate of change at any point will then be

$$\phi \frac{\partial C}{\partial t} = -\vec{v} \cdot \nabla C \tag{4}$$

Except for the expression of $C$ in some suitable normalized form, this additional equation imposes no further scaling restrictions except for the requirement of uniform or scaled porosity. But at any rate, if the determining hydrodynamic properties—the fluid density and viscosity—are the same in the displaced as in the displacing fluid, and if no special interfacial boundary conditions are imposed, then the interface will serve strictly as an indicator of integrated Darcy velocity. The dimensionally scaled model may thus, according to this, be operated with any pair of hydrodynamically similar fluids, or with any nonreactive tracer. How allowances may be made in modeling a porous matrix with variable properties or one containing fluids of varying density and viscosity will be indicated later. At this point, however, a different difficulty must be discussed.

## LIMITATIONS OF A MACROSCOPIC DESCRIPTION

In Darcy's law the variable $\vec{v}_{(x,y,z)}$ is used in a partial differential equation to represent the volumetric fluid velocity as a function of position. Such continuous functions, defined at all points of the space of interest, vary smoothly. In particular, the gradient of a harmonic function is smooth on a scale determined by the size and curvature of the boundaries. It is evident that even if it is assumed that the working fluid is really mathematically continuous and possesses a unique velocity at every point, one is unable to define any such velocity in the major part of the space included within the macroscopic or gross boundaries of the system here.

In fact, of course, Darcy's law is simply not valid on a microscopic scale; the actual fluid velocity in the pore space is not $(1/\phi)$ times the velocity $\vec{v}_{(x,y,z)}$ of Equation (1). Rather the flow within the interstices of the rock matrix may be described by the Navier-Stokes equation within the actual microscopic pore boundaries, together with the higher order and more detailed boundary conditions appropriate to the actual rock surfaces. Darcy's equation itself may be regarded as the law of flow of an imaginary fluid, moving in an imaginary space defined by the gross boundaries of the actual rock. This ectoplasmic fluid in its motion ignores the microscopic structure of the pore space; its velocity $\vec{v}_{(x,y,z)}$ is defined throughout the region of interest and represents an approximate average of the volumetric flux of the real fluid. The average in this case must be taken over a volume large enough so that statistical variations from one pore channel to another are evened out in the integration, but nevertheless small enough compared to the gross features of a sample so as not to mask the sorts of velocity variations which are of interest here.

The object of calling attention to these difficulties of interpretation is neither to advocate the regular use of the Navier-Stokes equation within porous media nor even to justify the Darcy formulation as appropriate in the macroscopic system. Rather the intention is to point out that the microscopic flow patterns must be considered when the gross or macroscopic displacement is under discussion.

Then consider the microscopic details of the following conceptual experiment inside a porous rock. Imagine that there is drawn around a given parcel of a fluid, whose velocity is given by the Navier-Stokes equation, a closed surface by which one may distinguish fluid inside from fluid outside. As the fluid moves through the rock, this surface, though suffering considerable distortion in shape, will remain inviolate. It will continue to enclose only the original quantity of fluid and that section of the rock matrix which it contained initially. Because the pore space is multiply connected, the moving imaginary interface will also become topologically entangled with more and more of the rock itself. The area of the surface, across which the displacing and displaced liquids are in contact, therefore, increases continually as the flow proceeds.

The microscopic histories of real fluid displacements depart considerably from the above description. The details of these actual situations depend markedly on the miscibility of the fluids. Two extremes may be noted, corresponding to the cases in which the displacing and displaced liquids are completely immiscible or miscible in all proportions. The intermediate situations in which one finds all real fluid-pairs contain the complications of both.

In the one extreme where the fluids are immiscible, a real interface does exist between them within the pore space. The motions of this interface are dominated by the forces of capillarity; its behavior consequently differs considerably from that of the imaginary interface described above. In particular, because the interfacial energy is an important part of the total free energy budget within the pore space, a continuous range of stable interfacial configurations of minimum area at a given pressure differential are generally isolated from their topological neighbors by dynamically unstable regions. Thus, flow which changes the average fluid content over a region—that is, displacement—occurs in discrete steps or Haines jumps. The particular pattern in which these are performed depends on the detailed fluid distribution as well as, perhaps, on the average rate at which the fluid is introduced. The pattern is also subject to a hysteresis—neither the microscopic nor the macroscopic picture is reversible. Two alternative approaches which have been taken in the description of the process of immiscible displacement and the scaling laws by which models of it may be constructed are of interest here.

The more common mathematical formulation of the immiscible displacement case is based on an extension of the Darcy equations of flow. This involves the definition of a pair of relative permeability functions giving the saturation-dependent proportionality constant between volumetric velocity and pressure drop. In order to consider explicitly the capillary effects, the relative permeability treatment has been refined by various investigators through the introduction of an additional dependent variable, the capillary pressure. Both the formulation of this approach and the development of dimensional modeling criteria for it have been discussed thoroughly in the literature; reference is made to the work of Rapoport and Leas (*1*) and Geertsma, Croes, and Schwartz (*2*).

A more ambitious approach which is based on the microscopic fluid behavior has also appeared in the literature recently. This work has been performed by E. E. Miller (*3*) and his collaborators among the soil physicists. It proceeds from the consideration of the Navier-Stokes equation within the pore space of a soil occupied by water and air. The detailed solution of the equation is not attempted; rather an inspectional analysis of the microscopic equations yields a set of similarity groups and reduced macroscopic variables. One of the principal features of this approach is the occurrence of two distinct characteristic lengths which are involved, respectively, in the microscopic and the macroscopic geometries. A further result is the concept that the liquid volume fraction (related to the saturation) and capillary conductivity (proportional to the effective permeability) are functionals or hysteresis functions of the relative pressure. This contrasts with the classical relative permeability approach in which saturation is regarded as an independent variable.

Though they contrast in many details, the two approaches quite naturally lead to some conclusions in common. For instance, the reduced or dimensionless time used as an independent variable in some of Miller's experiments

$$t_M = t\frac{\lambda\,\sigma}{\mu\,L^2} \tag{5a}$$

is almost the same as the dimensionless time derived by Mattax and Kyte (*4*) from the relative permeability approach. This variable

$$t_{MK} = t\sqrt{\frac{k}{\phi}}\,\frac{\sigma}{\mu\,L^2} \tag{5b}$$

contains instead of the microscopic characteristic length $\lambda$ the square root of the matrix permeability, $\sqrt{k}$, a number proportional to it. The fact that Mattax and Kyte's water-oil imbibition experiments can all be plotted as a function of this variable indicates further that Miller's microscopic approach has some validity for two-liquid as well as for gas-liquid systems.

In the alternative extreme of liquid behavior, the completely miscible situation, there are no fluid-fluid phase boundaries present in the pore spaces. In this case one may expect that the Navier-Stokes equation should be adequate to describe the detailed fluid velocities. An imaginary closed surface drawn around a fluid parcel could then be expected to follow the pattern discussed earlier. If the fluid inside the surface were distinguishable from that outside, one would find that after only a short interlude of flow, its distribution throughout the pore spaces would be a very rapidly varying function of position. In this hypothetical experiment, an attempt to find and follow the surface convolutions would be defeated by molecular diffusion. A boundary surface which moves with the fluid does not in fact permanently separate the fluids on each side; it is crossed continually by individual molecules in both directions. The sharpness of the border is thus progressively degraded.

## DIFFUSION AND FLOW

With an arbitrary volume element of fluid $V$, with surface $\Sigma$, (small enough to inhabit comfortably the pore space) the material transport equation is derived by equating the rate of change of tracer content within the volume to the sum of the surface integrals which represent the rates at which tracer is carried into the element by convection and by diffusion.

$$\frac{\partial}{\partial t}\int\int_V\int C_{(\vec r,t)}\,dV = -\int_\Sigma\int C_{(\vec r,t)}\,\vec v\cdot d\vec\Sigma + \int_\Sigma\int D\,\nabla\vec C\cdot d\vec\Sigma \qquad (6)$$

Here, the independent variables are position and time, $\vec r$ and t. The concentration of tracer material is represented by $C_{(\vec r,t)}$ and the fluid velocity by $\vec v_{(\vec r,t)}$. The molecular diffusion coefficient is denoted by $D$, while $V$ and $\Sigma$ measure respectively the volume and surface of the volume element. Then by use of the divergence theorem, the surface integrals on the right are transformed. An equality is obtained which, holding true for any arbitrary volume satisfying the initally stated specifications, is true everywhere within such a volume and may therefore be written as a differential equation:

$$\frac{\partial C}{\partial t} = -C\,\nabla\cdot\vec v - \vec v\cdot\nabla\vec C + \nabla\vec D\cdot\nabla\vec C + D\,\nabla^2 C \qquad (7a)$$

Interest here is focused on cases in which the fluid velocity is divergence free and in which also the diffusion coefficient is independent of position and thus the first and third terms on the right may be dropped:

$$\frac{\partial C}{\partial t} = -\vec v\cdot\nabla\vec C + D\,\nabla^2 C \qquad (7b)$$

In the same way in which the procedure was applied earlier to the Darcy system, dimensionless reduced variables are again introduced:

$$\left.\begin{aligned}\xi &= x/\lambda\\ \eta &= y/\lambda\\ \zeta &= z/\lambda\\ \tau &= tW/\lambda\end{aligned}\right\} \qquad (8a)$$

Here, $\lambda$ is a length which characterizes a particular real system, and $W$ is a characteristic scalar speed of the fluid at some point in it. When one ignores the transformation group of scalar multiplication of $C$ (since it may be considered to be in dimensionless form already) and defines a dimensionless variety $\vec w$ as the measure of $\vec v$ in units of $W$

$$\vec w = \vec v/W \qquad (8b)$$

Then the differential equation becomes

$$\frac{\partial C}{\partial \tau} = -\vec w\cdot\vec\nabla' C + \frac{D}{\lambda W}\nabla'^2 C \qquad (9)$$

Now the dimensionless velocity function $\vec w_{(\xi,\eta,\zeta)}$ is specified through the Navier-Stokes equation by its value and derivative on the boundary and by the boundary shape. For a given dimensionless velocity function $\vec w$, the concentration is a function not only of the dimensionless distance and time variables, but of the similarity group $\lambda W/D$. (It is convenient to use the reciprocal of the coefficient of the second term in Equation (9).) If $\lambda$ is a typical pore dimension, and $W$ the average liquid velocity, the dimensionless number $\lambda W/D$ measures the relative importance of convective to diffusive flux within the pore spaces. This number is important because some of the frontal dispersion results from the variation of liquid velocity. In addition to the molecular diffusion, and interacting with it, the variation of the fluid velocity from place to place tends also to disperse the boundary region between the fluids. Like the mixing of fluids in a chemical reactor, the mixing which occurs at the displacement boundary or front between two miscible liquids in a porous media consists of two stages. These result from the two distinct mixing phenomena corresponding to the two terms in the material transport equation. The convective distribution is greater if the number $\lambda W/D$ is large, for in this case the time constant for diffusion within a pore is long compared to the typical travel time for flow through it. On the other hand a small value of the number indicates that diffusion is virtually able to homogenize the contents of a pore in the time a bit of fluid takes to traverse it. This is evidently the situation in the lower range of flow rates in a typical oil sand, and in this case we can expect that the variation of velocity within a pore should not contribute to the overall frontal dispersion.

## DISPERSION AND ITS SCALING

Even at low values of $\lambda W/D$, however, larger scale variations of velocity between neighboring pores and groups of pores will markedly affect the frontal dispersion. If these average pore velocities are distributed randomly, that is, with no long-range autocorrelation in the sequences of fast and slow segments along the diffusion-averaged streamlines, and with no large-scale variation in a direction perpendicular to the flow of the average streamline velocities, then a mathematically tractable situation results. Under these conditions, as has been shown theoretically in various special cases by several workers, including Josselin DeJong (5), Scheidegger (6), Saffman (7), Aris (8), and Bear (9), the dispersion arising from velocity variations may be described by a diffusion equation in the macroscopic coordinates. Thus in a linear system (one in which the Darcy streamlines are everywhere parallel) in which the concentration is a function only of time and of the distance along the streamlines

$$\frac{\partial C_{(x,t)}}{\partial t} = -\frac{v}{\phi}\frac{\partial C}{\partial x} + \mathfrak{D}\frac{\partial^2 C}{\partial x^2} \qquad (10)$$

In this equation, $x$ is the distance measured along the flow lines from the inlet end, $v$ is the Darcy (volumetric) fluid velocity, and $\phi$ the porosity of the porous matrix. The coefficient $\mathfrak{D}$ is here descriptive of the gross frontal dispersion resulting from the interaction of the random velocity variations and the continuing molecular diffusion. Although it is similar is mathematical form to Equation (7*b*) discussed earlier, Equation (10) should be physically distinguished from it. The space to which Equation (10) refers is bounded by the gross boundaries of the sample and contains both sand grains and pore space. The functions in it, $C_{(x,t)}$ and $v_{(x)}$, represent smoothed values of the corresponding physical variables averaged over several pore spaces.

A slightly different approach to the method used above for the dimensional study of such differential equations is as follows. Seek from among the coefficients of the equation some characteristic values of the variables, such that a dimensionless equation written in terms of these

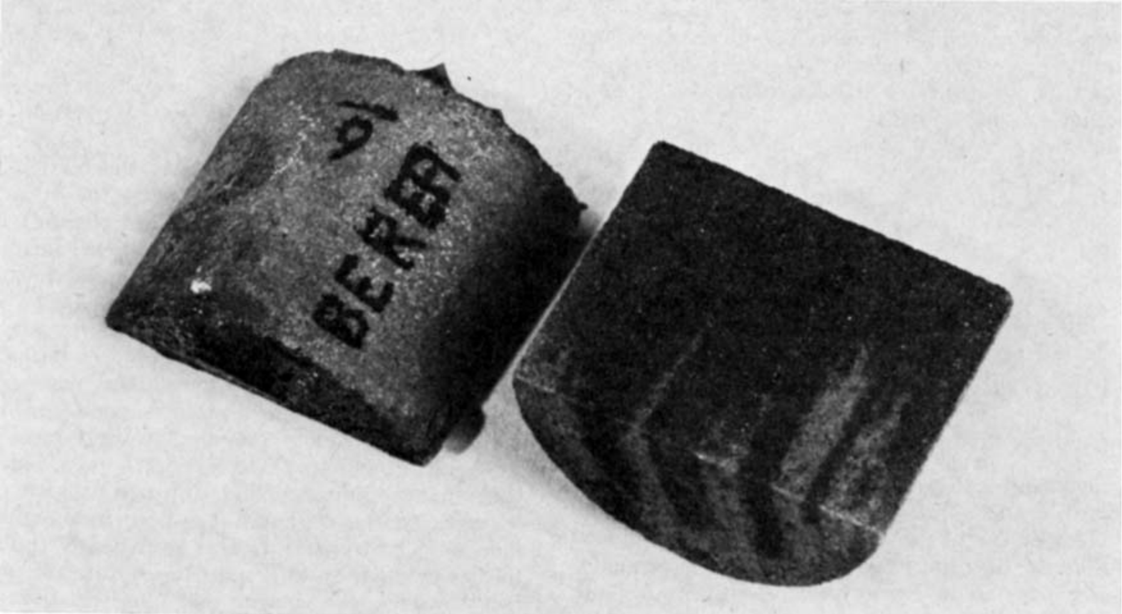

Fig. 1. Nonuniform miscible displacement.

natural units will be normalized. For instance, rewrite Equation (10) in terms of a dimensionless distance $\hat{x}$ and time $\hat{t}$ defined by

$$\hat{x} = \frac{x}{\phi\,\mathfrak{D}/v} \qquad \hat{t} = \frac{t}{\phi^2\mathfrak{D}/v^2} \qquad \hat{C}_{(\hat{x},\hat{t})} = C_{(x,t)} \tag{10a}$$

Transforming to the use of these new variables, Equation (10) becomes

$$\frac{\partial\hat{C}}{\partial\hat{t}} = -\frac{\partial\hat{C}}{\partial\hat{x}} + \frac{\partial^2\hat{C}}{\partial\hat{x}^2} \tag{10b}$$

For a step function input defined by

$$\left.\begin{aligned} \hat{C}_{H(\hat{x}<0,0)} &= 1 \\ \hat{C}_{H(\hat{x}>0,0)} &= 0 \end{aligned}\right\} \tag{10c}$$

the solution is

$$\hat{C}_{H(\hat{x},\,\hat{t})} = \frac{1}{2}\,\mathrm{erfc}\,\frac{1}{2}\left[\frac{x}{\sqrt{\hat{t}}} - \sqrt{\hat{t}}\right] \tag{10d}$$

Now the boundary conditions used here, (10*c*), are not strictly representative of the actual situation. The fact is that a real square front is usually inserted at the inlet face of the system rather than in its interior. Thus a more realistic boundary condition might be written

$$\left.\begin{aligned} \hat{C}_{S(0,\hat{t}>0)} &= 1 \\ \hat{C}_{S(\hat{x}>0,0)} &= 0 \\ \hat{C}_{S(\hat{x}\to\infty,\hat{t})} &= 0 \end{aligned}\right\} \tag{10e}$$

In this event the solution of Equation (10*b*) is

$$\hat{C}_{S(\hat{x},\,\hat{t})} = \frac{1}{2}\,\mathrm{erfc}\,\frac{1}{2}\left[\frac{\hat{x}}{\sqrt{\hat{t}}} - \sqrt{\hat{t}}\right] + \frac{1}{2}\,e^{\hat{x}}\,\mathrm{erfc}\,\frac{1}{2}\left[\frac{\hat{x}}{\sqrt{\hat{t}}} + \sqrt{\hat{t}}\right] \tag{10f}$$

This differs from $\hat{C}_H$ the idealized solution above, by the addition of a second term. This second term may be thought of as an image term representing the concentration contribution of a virtual front which travels in the opposite direction to the flow. The decrease of the error function complement is stronger than the exponential factor; the entire second term thus fades out of sight as $\hat{x}$ increases beyond unity. A return to the approximate solution (10*d*) is then justified. In terms of the familiar dimensional variables, the geometry of this may be described by the thickness of a transition zone. If $\Delta X$ represents the distance between the isoconcentration planes $C = 0.9$ and $C = 0.1$

$$\Delta X = 3.62\sqrt{\mathfrak{D}T} = 3.62\sqrt{\frac{\mathfrak{D}\,\phi\,X}{v}} \tag{11}$$

where $X$ is the distance a sharp front has proceeded in the time since its initiation. This abbreviated form is a valid approximation to the actual solution (involving the second term mentioned above) everywhere except close to the inlet face. Further away than a few of the natural distance units $\phi\mathfrak{D}/v$ the second term in (10*f*) is negligible (see reference *10*).

A great many experiments have been interpreted on the basis of Equation (11). The usual method of interpretation has been the calculation of the effective dispersion coefficient $\mathfrak{D}$ from concentration measurements at the outflow face of laboratory flow systems. Some words of caution are needed here: both the experimenter and the theorist awaiting his results are prone to accept upon insufficient evidence the assumptions made in deriving Equation (10). The fact is that they do not apply to all porous systems. An output concentration curve which coincides, within experimental error, with the curve of the error function does not guarantee that all of the enumerated conditions of randomness of fluid velocity are met.

It is not only among unconsolidated packs that such anomalies occur, but also in naturally cemented materials. Figure 1 gives evidence of such a situation. This is a section of a ¾-in diam. core plug of the much used sample material, Berea sandstone. A miscible flood experiment was performed using catalyzed styrene monomer as the working fluid. After saturation of the pore space with clear fluid, a sharp front of the same material dyed with methyl violet was started in from the far end face. When about half of a pore volume of the clear plastic had been displaced, flow was stopped. After polymerization of the styrene the core plug was sectioned and polished.

Even such flow irregularities as these, unless the channels are correlated over long distances, without effective diffusive or convective interchange between them will permit of the simplified treatment given by Equation (10) for wide transition zones. But the possible existence of extensive correlated velocity zones or channels cannot be discounted a priori for either flow samples, laboratory models, or reservoirs.

If such channels are not of major importance, one may return to the diffusion type of differential Equation (10) of material transport which uses the Darcy velocity and a smoothed concentration function. In recognition of the experimentally observed difference between the effective dispersion coefficients (*11* and *12*) parallel and perpendicular to flow, one may extend (10) to three dimensions to read

$$\frac{\partial C}{\partial t} = -\frac{v}{\phi}\frac{\partial C}{\partial x_1} + \mathcal{D}_L \frac{\partial^2 C}{\partial x_1^2} + \mathcal{D}_T\left(\frac{\partial^2 C}{\partial x_2^2} + \frac{\partial^2 C}{\partial x_3^2}\right) \tag{12}$$

Here, $x_1$ is measured in the direction of the Darcy velocity $v$, and $x_2$ and $x_3$ in two mutually perpendicular directions transverse to it. The coefficients of longitudinal and transverse dispersion are respectively $\mathcal{D}_L$ and $\mathcal{D}_T$.

Two geometrically similar flow systems, which are both described by the reduced Darcy Equation (1′) and by Equation (12), are not dimensionally scaled unless further requirements are met. This results from the additional terms in Equation (12) which were not present in the simply convected concentration Equation (4). Equation (12) may be put into dimensionless form by the use of a characteristic macroscopic length $A$ and rate $W$ and by using the new independent variables:

$$\xi_i = x_i/A, \quad \tau = tW/A$$

Then

$$\frac{\partial C}{\partial \tau} = -\frac{U}{\phi}\frac{\partial C}{\partial \xi_1} + \frac{\mathcal{D}_L}{AW}\frac{\partial^2 C}{\partial \xi_1^2} + \frac{\mathcal{D}_T}{AW}\left(\frac{\partial^2 C}{\partial \xi_2^2} + \frac{\partial^2 C}{\partial \xi_3^2}\right) \tag{13}$$

is obtained. Here $U$ is $v/W$, the dimensionless volumetric velocity which occurred in the reduced form of Darcy's Equation (1′).

From Equation (13) one sees that two dimensionless groups must be considered. These are $\mathcal{D}_L/AW$ and $\mathcal{D}_T/AW$. If flow models are to be dimensionally scaled the values of these two numbers must not vary from model to model. The preservation of these dimensionless groups would not be difficult were it not for a complicating factor regarding the effective dispersion coefficients. Even in well-behaved fluid-matrix systems whose dispersion behavior can be described by the differential Equation (12) or (13), the coefficients $\mathcal{D}_L$ and $\mathcal{D}_T$ depend on the velocity of the fluid. With reference to the microscopic picture discussed above, the ratio of the effective dispersion coefficient to the molecular diffusion coefficient in the fluid may be expressed as a Taylor expansion in powers of the dimensionless parameter $\lambda W/D$

$$\frac{\mathcal{D}_{L;T}}{D} = \frac{1}{\phi F} + a_{1L;T}\frac{\lambda W}{D} + a_{2L;T}\left(\frac{\lambda W}{D}\right)^2 + \ldots \tag{14}$$

The first term of the expansion is related simply to the electrical formation resistivity factor which may be independently measured. The coefficients $a_{1L;T}$ and $a_{2L;T}$ are descriptive of the microscopic structure of the matrix. Their numerical values are somewhat arbitrary, at least until a reasonable general rule is enacted for the measurement of the characteristic microscopic length $\lambda$[*]. Further, the coefficients $a_{2L}$ and $a_{2T}$ are evidently quite small for porous packs of convex particles judging from the experimental curves published in the literature.

One finds that the product $a_{1L}\lambda$ is large enough so that for common reservoir liquids moving at only moderately high rates

$$\frac{a_{1L}\,\lambda W}{D} >> \frac{1}{\phi F}$$

In this case, note that Equation (14) looks like

$$\mathcal{D}_L \approx \lambda\, a_{1L}\, W \tag{14a}$$

With reference to the modeling requirement stated above in terms of the dimensionless group $\mathcal{D}_L/AW$, one sees that if $\mathcal{D}_L$ is approximated by (14*a*) in both models, the velocity condition is removed and the dimensionless groups degenerate to $\lambda a_{1L}/A$ in both models. This leads to the relative independence of fractional mixing zone length on the diffusion coefficient or on the displacement velocity over a large range of velocities. Observation of this rate independence dominated many early model experiments. It led to various erroneous beliefs that the breakthrough recovery and required slug size could be inferred from semi-model experiments in which the longitudinal dispersion behavior was not scaled. The consequences of ignoring the dispersion scaling requirements vary with the circumstances: the shape of the reservoir and disposition of the wells, the displacement rate, and a further effect which is yet to be considered here.

Transverse dispersion is not of direct importance in the longitudinal growth of a transition zone; but it significantly affects the displacement whenever the concentration front is not perpendicular to the Darcy velocity. Such a situation can occur both because of the macroscopic geometry in which the Darcy streamlines are curved and as a result of nonuniformities of fluid or rock properties.

## FLOW NONUNIFORMITIES

The simplest treatment of liquid displacement was based on Darcy's law with constant parameters. It evidently does not produce obvious errors to consider Darcy's law to be just as valid when the parameters vary. When Equation (1) is rewritten taking account explicitly of the gravitational forces

$$\vec{\nabla} p + \frac{\mu}{k}\vec{v} + \rho\vec{G} = 0 \tag{15}$$

The symbols here are familiar except for those in the last term. The fluid density, which along with $\mu$ and $k$ will be considered a function of position, is represented by $\rho$. The vector $\vec{G}$ is a constant directed upward and having the scalar magnitude g, the acceleration due to gravity. This equation, the incompressibility relation by which the divergence of $\vec{v}$ vanishes, and the macroscopic material transport equation for the concentration (12), determine the response of the flow system to the initial and boundary conditions.

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[*] Values of the products $a_{1L}\lambda$ and $a_{1T}\lambda$ appropriate to various kinds of materials may be calculated from the results presented in references *11* and *12*.

A convenient mathematical technique by which the dependence of the system on nonuniformities of $\mu$, $\rho$, and $k$ may be examined is the vector differentiation of (15). When the curl of this equation is taken, the fluid pressure is dropped as a variable. The remaining terms lead to

$$\frac{\overrightarrow{\nabla\mu}}{k} \times \vec{v} - \frac{\mu}{k^2} \vec{\nabla} k \times \vec{v} + \frac{\mu}{k} \nabla \times \vec{v} + \vec{\nabla}\rho \times \vec{G} = 0$$

The variation of permeability is fixed in the Eulerian coordinate frame, but the density and viscosity depend on the composition of the fluid. Accordingly they are written as functions of $C$ and abbreviated

$$\left.\begin{aligned} \rho' &\equiv \frac{d\rho}{dC} \\ \mu' &\equiv \frac{d\mu}{dC} \end{aligned}\right\}$$

These definitions lead to an expression for the vorticity or curl of the velocity:

$$\nabla \times \vec{v} = \left( \frac{\mu'}{\mu} \vec{v} + \frac{\rho' k \vec{G}}{\mu} \right) \times \vec{\nabla} C + \nabla(\overrightarrow{\ln} k) \times \vec{v} \tag{16}$$

Note that in the simple Darcy situation in which viscosity and density do not depend on concentration, and in which there are no permeability variations, each of the terms of this equation vanishes separately.

Repeating the use of the characteristic length $A$ and characteristic Darcy velocity $W$, this equation is put into the dimensionless form:

$$\nabla' \times \vec{U} = \left( \frac{\mu'}{\mu} \vec{U} + \frac{\rho' k \vec{G}}{\mu W} \right) \times \vec{\nabla} C + \frac{1}{k} \vec{\nabla}' k \times \vec{U} \tag{16a}$$

The dimensionless parameters here are the logarithmic derivative of viscosity with concentration, $\frac{1}{\mu}\frac{d\mu}{dC}$, the product, $\frac{\rho' k g}{\mu W}$, and the gradient of the logarithm of $k$. Note that in the second of these, permeability must be measured in its centimeter-gram-second unit, the square centimeter if the terms are to be compared numerically. Thus for two flow systems to be dimensionally scaled models of each other, they must not only be geometrically similar and satisfy the dispersion-similarity requirements cited above, they must also be similarly oriented with respect to the body force field in which they find themselves, and the three dimensionless parameters of Equation (16) must have equal values in the two systems. If the two dimensionless groups for dispersion, $\mathcal{D}_L/AW$ and $\mathcal{D}_T/AW$, are the same for both systems and the initial flow conditions similar, then the reduced concentration gradient will be represented by the same function of dimensionless distance and time.

The vorticity of the Darcy velocity may also be studied analytically by the use of Equation (15) or (16). It gives the convective contribution to the local angular velocity of the isoconcentration lines, an important quantity for calculation of the growth of instability fingers.

If the determining vector

$$\left( \frac{\mu'}{\mu} \vec{v} + \frac{\rho' k \vec{G}}{\mu} \right)$$

lies parallel or anti-parallel to the concentration gradient, and if the gradient of the permeability is zero, then the curl of the velocity vanishes despite the sign of the viscosity and density gradients. This dynamic balance is, however, unstable in the unfavorable case when the two vectors point in the same direction. In this case, a slight departure from parallelism will give rise to a positive vorticity which will rotate the isoconcentration surfaces, aggravating the original disturbance. The dynamics of this process, if the appropriate similarity groups are carefully controlled, may be studied in scale models. Especially important in such modeling are the reduced concentration distribution—the width and shape of the transition zones in the different parts of the model—and the value of the transverse dispersion group, $\mathcal{D}_T/AW$.

Fingering, the evidence of frontal instability, generally originates from inhomogeneities in the porous material itself—from randomly distributed sources of vorticity at which the third term of Equation (16) $\overrightarrow{\nabla(\ln k)} \times \vec{U}$ becomes appreciable. For complete dimensional scaling, the permeability of the model must be proportional to that of the prototype at corresponding points. This is a stringent requirement which cannot be satisfied in detail simply because the properties of the prototype reservoir are never known very intimately. Instability fingers will not start at the same reduced places or at the same reduced times in the field as in the model. This unknown factor must be recognized in the engineer's interpretations.

## CONCLUSION

The natural appeal of models as cozy representations of more inconveniently sized physical systems can lead to the uncritical acceptance of at least the more intriguing ones as complete analogs of their geometrical prototypes. The major precaution in the use of models is that the physically proper equations should be used. In the discussion above, attention has been fixed on the physical phenomena underlying these equations. These equations are mathematical expressions of the physical processes occurring in both prototype and model.

In view of the requirement that even for a model study the mathematics must be followed closely, one might question the use of models at all. If all of the equations of flow must be utilized and the values of the various similarity groups controlled, why not proceed with a numerical analysis based on the known mathematical situation? Furthermore, do not these arguments apply with particular strength against the various semi-models which are based on only an incomplete mathematical description of the flow?

They do to some extent. With the rapid technological progress in digital computation, numerical analysis becomes more and more attractive for a variety of scientific and engineering problems. Models and semi-models continue to be used for fluid displacement studies, however. Their value lies especially in elucidating the behavior of prototypes with unsymmetrical boundaries and many wells. In these cases a sufficiently detailed numerical analysis is very demanding of machine time and storage facilities. Various general studies, such as that of the long-term growth of particular instability patterns and the ways in which they are influenced by dispersion, may also be conveniently studied by flow models.

One should remember that fluid displacement includes some rather complex phenomena, and that neither of the analytical approaches discussed here can claim infallibility. In fact, in view of the incompleteness of the knowledge about many of the processes going on in porous media, it would probably be wise to regard model flow systems and numerical analysis as complementary rather than competing techniques for the prediction of reservoir displacement behavior.

**ACKNOWLEDGMENT**

The author wishes to express his appreciation to the management of Socony Mobil Oil Company, Inc., for permission to publish this paper, and to John Masse for the preparation of the core sample displayed in the photograph.

## NOTATION

$(x, y, z) = \vec{r} =$ position variables, vector
$t$ = time
$A, L$ = macroscopic characteristic lengths
$\vec{v}$ = velocity vector, Darcy velocity vector
$\vec{U}, \vec{w}$ = dimensionless velocity vector
$W$ = characteristic scalar flow speed
$C$ = concentration
$p_{(\vec{r},t)}$ = pressure function

$p_i, p_o$ = inlet and outlet pressure
$k$ = permeability
$F$ = electrical formation resistivity factor
$D$ = diffusion coefficient
$\mathcal{D}$ = effective dispersion coefficient (subscripted $T$ or $L$ for transverse or longitudinal)
$a_{1T}, a_{1L}; a_{2T}, \ldots$ = coefficients in expansion of $\mathcal{D}_T$ and $\mathcal{D}_L$
$\vec{G}$ = acceleration due to gravity, vector directed vertically upward

### Greek Letters

$\xi, \eta, \zeta$ = dimensionless position variables
$\tau$ = dimensionless time
$\lambda$ = microscopic characteristic length
$\phi$ = porosity
$\mu$ = fluid viscosity
$\rho$ = fluid density
$\sigma$ = interfacial tension
$\Sigma$ = surface of volume element $V$

### Superscript

$\wedge$ = dimensionless variable, as in $\hat{p}$

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*Manuscript received September 18, 1962; revision received January 14, 1963; paper accepted January 14, 1963. Paper presented at A.I.Ch.E. Denver meeting.*

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# Continuous Propagation of Microorganisms

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**Microorganisms undergo a cycle of maturation during their life-span in a proliferating culture. This gives rise to a distribution of degrees of cellular maturation in the culture. The gross rate of any metabolic process carried on by the culture is therefore an average over all degrees of maturation present.**

**Equations embodying the life-cycle concept are developed for cultures of organisms which proliferate by binary fission and do not form spores. The equations are solved for the age distribution in the special case of a continuous, steady state fermentation. Expressions for calculating gross metabolic rates and cell size distributions are derived. Finally, the stability of the steady state fermentor is discussed.**

Monod (*19*), Novick and Szilard (*22*), Maxon (*18*), Herbert et al. (*11*), and others have developed theories of microbial propagation which, in some cases, give accurate quantitative representation of certain aspects of growth phenomena. The theories mentioned are based on two postulates.

1. The growth rate of a culture is proportional to the rate of consumption of substrate.
2. The specific growth rate of a culture is some function of the substrate concentration.

These two postulates obviously apply only to cases where the availability of one substrate limits the growth rate.

The two postulates may be expressed mathematically as

$$R = -\beta R_s \tag{1}$$

$$R/C = F(C_s) \tag{2}$$

in which $C$ is the microbial concentration, $C_s$ is the substrate concentration, and $\beta$ is the yield coefficient. It is usually assumed that the function $F(C_s)$ is of the Michaelis-Menten form [Monod (*19*)]

$$F(C_s) = \frac{\mu_m C_s}{K_s + C_s} \tag{3}$$

where $\mu_m$ and $K_s$ are constants, the former being the maximum specific growth rate.